Article

Uniform Resolvent Estimates for Schrödinger Equations in an Exterior Domain in $\mathbb{R}^2$ and Their Applications to Scattering Problems

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Abstract

Uniform resolvent estimates for stationary Schrödinger and dissipative wave equations in a two-dimensional exterior domain are reported. The smoothing estimate for the corresponding evolution equations and the principle of limiting amplitude for dissipative wave equations are also obtained.

1. Introduction and Results

Recently, Mochizuki proved the uniform resolvent estimate for stationary magnetic Schrödinger equations in $\mathbb{R}^N$ or $\Omega \subseteq \mathbb{R}^N$ (the exterior domain of a star-shaped obstacle) with $N \geqq 3$ ([15] and [16]). He also obtained smoothing estimates for the corresponding evolution equations. Mochizuki's proofs were based on Hardy-type inequalities related to the radiation conditions (= referred to as Mochizuki's inequality).

However, the corresponding result in a two-dimensional exterior domain was left as a problem for future study.

This paper generalizes Mochizuki's inequality and presents uniform resolvent estimates in the exterior domain in $\mathbb{R}^2$. Details of the derivations and their proofs will be published elsewhere.

Assume that the number of dimensions $N \geqq 2$. Let Ω be a whole space $\mathbb{R}^N$ or an exterior domain of a star-shaped obstacle in $\mathbb{R}^N$ satisfying $0 \notin \Omega$. Here, Ω is star-shaped if $\left(\frac{x}{r}, n\right) \leqq 0$ for any $x \in \Omega$ and for any unit outer-normal n of $\partial\Omega$. Assume that the function u is a solution to a Schrödinger equation of the form

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$$\left(-\Delta + V(x) - \kappa^2\right) u(x) = f(x), \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega, \qquad (1)$$

where $V(x)$ is a real valued $C^1(\Omega)$ – function and $\kappa \in \mathbb{C}$ denotes a spectral parameter.

Let $r = |x|$ and define the operator $D_r^{\pm}$ as

$$D_r^{\pm} u = u_r + \frac{N-1}{2r} u \mp i\kappa u \quad (\pm \Im\kappa \geqq 0),$$

where $u_r = \left(\nabla u, \frac{x}{r}\right)$ ($(\cdot, \cdot)$ denotes the usual L^2–norm).

These operators are introduced in Ikebe-Saito [3] and Mochizuki [12]. We define the weighted L^2–norm and the weighted L^2–space L^2_w by

$$||u||^2_w = \int_\Omega w(x) |u(x)|^2 dx, \qquad L^2_w = \{u \,|\, ||u||_w < \infty\}$$

for some non-negative weight function $w(x)$

The weight function adopted in this paper must satisfy the following condition.

Condition 1.1. *$\varphi = \varphi(r)$ is a non-negative L^1 function of r satisfying $\varphi_r \leqq 0$.*

Example 1.1. *The two candidate functions are $(1+r)^{-1-\delta}$ $(\delta > 0)$ and $(e + r)^{-1} \{\log(e+r)\}^{-1-\delta}$ $(\delta > 0)$.*

Assumption 1.1. *We assume that $|V(x)| \leqq Cr^{-2}$ and $(rV(x))_r \leqq 0$ for some $C > 0$.*

These considerations lead to the following theorem:

Theorem 1.1. *Let u be a solution of (1) with radiation condition $||D_r^{\pm} u||_\varphi < \infty$. If the function $V(x)$ satisfies Assumption 1.1, and φ satisfies*

$$\inf_{r \geqq r_0} \left(-\frac{r\varphi_r}{\varphi}\right) \geqq \frac{1}{2} \qquad (r_0 = dist(x, \partial\Omega)),$$

then

$$|\kappa|^2 ||u||^2_\varphi + ||u||^2_{-\frac{\varphi_r}{r}} + ||D_r^{\pm} u||^2_\varphi \leqq C ||f||^2_{\frac{r^2}{\varphi}}. \qquad (2)$$

holds for some constant $C > 0$ independent of κ.

Remark 1.1. (*i*) *With regard to [15] and [16], this theorem is meaningful if and only if* $N = 2$*. If* $N \geqq 3$*, Mochizuki's result is sharper than our result.*

(*ii*) *If the potential function* $V(x)$ *takes the form*

$$V(x) = \frac{V_2}{r^2} + V_3(x)$$

for some constant $V_2 \geqq \frac{1}{4}$ *and for some function* $V_3(x)$ *satisfying Assumption 1.1, the usual resolvent estimate*

$$|\kappa|^2 ||u||^2_{(1+r)^{-1-\delta}} \leqq C||f||^2_{(1+r)^{1+\delta}} \quad (\delta > 0) \tag{3}$$

is easily proved for $N = 2$*.*

(*iii*) *Especially, the following inequality is obtained from* (2)*:*

$$||u||^2_{-\frac{\varphi_r}{r}} \leqq C||f||^2_{\frac{r^2}{\varphi}}.$$

The left-hand side of this inequality is independent of the spectral parameter κ*. Therefore, we refer to this estimate as the uniform resolvent estimate. For the optimal result under* $N \geqq 3$*, the reader is referred to [15] and [16].*

(*iv*) *Unlike* (3)*, this inequality violates the dual relationship between the weight functions of both sides of* (2)*, i.e.,* φ *and* $\frac{r^2}{\varphi}$*.*

(*v*) *For example, taking* $\varphi(r) = (1+r)^{-1-\delta}$ $(\delta > 0)$*, inequality* (2) *gives*

$$|\kappa|^2 ||u||^2_{(1+r)^{-1-\delta}} + ||u||^2_{(1+r)^{-3-\delta}} + ||D_r^{\pm} u||^2_{(1+r)^{-1-\delta}} \leqq C||f||^2_{(1+r)^{3+\delta}}.$$

Throughout the whole space $\Omega = \mathbb{R}^N$, the usual (local) resolvent estimate (3) with $C = C(\kappa) > 0$ has been established by Kuroda [7] and Agmon [1] for $V \equiv 0$. The global version of this estimate (in which the constant C is independent of κ) has been proven by Mochizuki [13]. The uniform resolvent estimate and its related smoothing estimate in $\mathbb{R}^N$ have also been demonstrated by Yafaev [24] for the case $V \not\equiv 0$.

To prove Theorem 1.1, we rely on Hardy-type inequalities related to radiation conditions:

Proposition 1.1. *Assume that $N \geqq 1$, and let v be a function in $C_0^\infty(\Omega)$. Let a satisfy $a \in (0,1]$ and $\phi = \phi(r) \in C^1(\Omega)$ be some weight function. Then the following two inequalities hold:*

$$(i) \qquad \left|\left| v_r + \frac{N-1}{2r} v \right|\right|_\phi^2 \geqq ||v||_{h_a}^2, \qquad (4)$$

$$(ii) \quad \left|\left| D_r^\pm v \right|\right|_\phi^2 \geqq \pm\Im\kappa ||v||_{\frac{2a\phi}{r}}^2 + ||v||_{h_a}^2, \qquad (5)$$

$$\text{where} \ \ h_a(r) = -\frac{a\phi_r(r)}{r} - \frac{a(a-1)\phi}{r^2}.$$

Remark 1.2. (i) *Mochizuki [15], [16] established the above inequality* (5) *for a specific weight function (see [15] Lemma 9).*

(ii) *Inequalities* (4) *and* (5) *also hold under the following change of operator*

$$\nabla v \ \rightarrow \ \nabla_m v \equiv \nabla v + im(x)v, \qquad v_r \ \rightarrow \ \nabla_m v \cdot \frac{x}{r},$$

where $m(x) = (m_1 x), m_2(x), \cdots, m_N(x))$ and each $m_j(x)$ $(j = 1, 2, \cdots, N)$ is a real-valued C^1 – function. The operator ∇_m appears in studies of the magnetic Schrödinger operator. In [15] and [16], inequality (5) *is established for this operator when $N \geqq 3$.*

Here, we present a rationale for the proof of Theorem 1.1 rather than a rigorous analysis (for more precise discussion, see the forthcoming paper). The non-negative terms identified by the previous authors (see e.g., [3], [12], [13], [14], [15], [16], [18] and [20]) are not omitted in our arguments, but are estimated more precisely using the inequalities in Proposition 1.1. To estimate the term $||u||_{-\frac{\varphi_r}{r}}^2$, we utilize (5) with $a = 1/2$. To estimate the term $\pm\Im\kappa ||u||_{\frac{\varphi}{r^2}}^2$ $(\pm\Im\kappa \geqq 0)$, we apply (4) with $a = 1$. Now, we can compensate the term

$$\frac{(N-1)(N-3)}{4r^2}$$

which remained negative, and therefore inestimable for $N = 2$ in previous analysis.

Once Theorem 1.1 is established, the smoothing estimates[2] for the corresponding evolution equations can be deduced from [5], [15] and [16].

Let $L = -\Delta + V(x)$ be the Schrödinger operator and consider the following equations:

$$iu_t - Lu = 0, \qquad u(0) = f \in L^2(\Omega), \tag{6}$$

$$iu_t - \sqrt{L + m^2}u = 0, \qquad u(0) = f \in L^2(\Omega), \tag{7}$$

$$u_{tt} + (L + m^2)u = 0, \quad u(0) = f_1 \in \dot{H}^1(\Omega), \ \ u_t(0) = f_2 \in L^2(\Omega), \tag{8}$$

where $m \geqq 0$ and $\dot{H}^1$ is the completion[3] of C_0^∞ with respect to the norm $||\nabla\cdot||^2_{L^2}$.

Theorem 1.2. *Assume that $N \geqq 2$, $V(x)$ satisfies Assumption 1.1 and $V \geqq 0$. Let the weight function φ satisfy*

$$\frac{1}{2} \leqq \frac{-r\varphi_r}{\varphi} \leqq C$$

for some $C > 0$. Then for the function defined by

$$h(r) = \sqrt{\frac{-\varphi_r}{r}}$$

the following inequalities hold:

(i) If u is a solution of (6) or (7), then

$$\left| \int_0^{\pm\infty} ||hu(t)||^2_{L^2} dt \right| \leqq C||f||^2_{L^2}.$$

(ii) If u is a solution of (8), then

$$\left| \int_0^{\pm\infty} ||hu_t(t)||^2_{L^2} dt \right| \leqq C||f||^2_{E},$$

where $E = \dot{H}^1 \times L^2$.

Remark 1.3. *Under the condition $V \geqq 0$, the Schrödinger operator L has a self-adjoint extension (the Friedrichs extension; see [15] and [16] for details).*

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[2] In general, if the solution becomes smoother than the initial data for some differential equation, we say that a smoothing effect occurs. The inequality which means this effect is called the smoothing estimate.

[3] The space so that the Cauchy sequence converges with respect to this norm.

Finally, consider the initial boundary value problem for the dissipative wave equation of the form

$$\begin{cases} w_{tt} - \Delta w + b(x)w_t = f(x)e^{-i\kappa t}, & (x,t) \in \Omega \times \mathbb{R}, \\ w\big|_{t=0} = w_t\big|_{t=0} = 0, \ \ x \in \Omega, & w = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R} \end{cases} \tag{9}$$

and its stationary equation

$$\left(-\Delta - i\kappa b(x) - \kappa^2\right) u(x) = f(x), \ \ x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega, \tag{10}$$

where $f \in L^2_{\frac{r^2}{\varphi}}$ and the function $b(x)$ satisfies

$$(B) \quad |b(x)| \leqq b_0 \varphi(r) r^{-1}$$

for some φ satisfying Condition 1.1 and for some $b_0 \in [0, 1/3)$.

The uniform resolvent estimate for (10) is proven identically to Theorem 1.1:

Theorem 1.3. *Assume $N \geqq 2$ and inequality (B) above. Assume also that the function φ satisfies*

$$\inf_{r \geqq r_0} \left(-\frac{r\varphi_r}{\varphi} \right) > \frac{1 + b_0}{2(1 - 3b_0)} \qquad (r_0 = dist(x, \partial\Omega)).$$

Then if (10) is solved with radiation condition $||D_r^{\pm} u||^2_{\varphi} < \infty$, the following inequality holds

$$|\kappa|^2 ||u||^2_{\varphi} + ||u||^2_{-\frac{\varphi_r}{r}} + ||D_r^{\pm} u||^2_{\varphi} + \int_{\partial\Omega} \{-(x,n)\} |u_n|^2 dS \leqq C||f||^2_{\frac{r^2}{\varphi}} \tag{11}$$

where n denotes the unit outer normal to the boundary $\partial\Omega$.

By using this theorem, the following theorem (on the principle of limiting amplitude for dissipative wave equations) is proved as an extension of Mizohata-Mochizuki's result [11].

Theorem 1.4. *Under the conditions of Theorem 1.3, and given solution w of (9) and u of (10), we have*

$$||w(\cdot, t)e^{i\kappa t} - u(\cdot)||^2_{\frac{\varphi}{r^2}} \to 0 \quad (t \to +\infty).$$

Remark 1.4. *(i) Mizohata-Mochizuki [11] treated the case $\Omega \equiv \mathbb{R}^3$ and non-negative $b(x)$ satisfying $b(x) \leqq Cr^{-3}$ for $r \geqq R > 0$ without imposing a smallness condition. In Theorem 1.4, the decay condition at infinity is relaxed in $\Omega \subseteq \mathbb{R}^N$ with $N \geqq 2$ and $|b(x)| \leqq b_0(1+r)^{-2-\delta}$ if we select $\varphi = (1+r)^{-1-\delta}$ $(\delta > 0)$.*

(ii) Under the conditions of Theorem 1.3, the principle of limiting absorption for the operator pencil $L(\kappa) = -\Delta - i\kappa b(x) - \kappa^2$ follows; namely,

$$\lim_{\Im\kappa \to \pm 0} L(\kappa)^{-1} f \in L^2_{\varphi}$$

for any $f \in L^2_{r^2\varphi^{-1}}$ ([18]). Moreover, the scattering states also exists ([13], [19], [20]).

For the proof of Theorem 1.4, we can follow the argument by Roach-Zhang [22]. If it states in more detail, some energy estimates for (10) are derived from inequality (11). These estimates are useful to prove Theorem 1.4. In the final stage of the proof, we adopt the argument in [11].

Other results about the dissipative wave equations (9) with $f \equiv 0$ are treated in references [17], [4] and [6].

The remainder of this section is devoted to relevant terms and their backgrounds.

Given a (possibly unbounded) operator L, if the operator $L-\xi$ has a bounded inverse for some $\xi \in \mathbb{C}$, the resolvent of L is defined by $(L-\xi)^{-1}$. For the eigenvalue problem $(L-\xi)u = f$, it is important to obtain the estimate $|\xi| ||u||_{w_1} \leqq C||f||_{w_2}$ for weight functions w_1 and w_2 in mathematical scattering theory. This problem is equivalent to estimating $|\xi| || (L-\xi)^{-1} f||_{w_1} \leqq C||f||_{w_2}$, which we call the resolvent estimate. Widely used weight functions for the Helmholtz equation, where $L = -\Delta$, are $w_1 = (1+r)^{-1-\delta}$, $w_2 = w_1^{-1} = (1+r)^{1+\delta}$ for some $\delta > 0$. See [7], [14] or [24].

The radiation condition proposed by Sommerfeld [23] and Rellich [21] is a class of boundary conditions at infinity that guarantees the unique solutions to the Helmholtz equation (see also [14]).

If the operator depends on the spectral parameter $\xi \in \mathbb{C}$ as $A(\xi) = \sum_{j=0}^{n} \xi^j A_j$ for some operators A_j, $A(\xi)$ is called an operator pencil (see for example, reference [9]). An example is the quadratic operator pencil $L(\kappa)$ in Remark 1.4 (ii).

2. Proof of Proposition 1.1 and Related Inequalities

In this section, we prove Proposition 1.1 and the related inequalities derived by the same methods.

[Proof of Proposition 1.1.] (i) By direct calculation, we obtain

$$\begin{aligned} 0 &\leqq \phi\left|v_r + \frac{N-1}{2r}v - a\frac{v}{r}\right|^2 \\ &= \phi\left|v_r + \frac{N-1}{2r}v\right|^2 - \nabla\cdot\left(\frac{a\phi}{r}|v|^2\frac{x}{r}\right) + \frac{a\phi_r}{r}|v|^2 + \frac{a(a-1)\phi}{r^2}|v|^2. \end{aligned}$$

Integrating both sides of this inequality in Ω, we obtain the desired results (4).

(ii) By a similar procedure, we also obtain

$$\begin{aligned} 0 &\leqq \phi\left|D_r^{\pm}v - a\frac{v}{r}\right|^2 \\ &= \phi|D_r^{\pm}v|^2 - \nabla\cdot\left(\frac{a\phi}{r}|v|^2\frac{x}{r}\right) \mp \Im\kappa\frac{2a\phi}{r}|v|^2 + \frac{a\phi_r}{r}|v|^2 + \frac{a(a-1)\phi}{r^2}|v|^2. \end{aligned}$$

Integrating both sides in Ω yields the desired results (5). □

Remark 2.1. (i) *As mentioned in Remark* 1.2(i)*, Mochizuki proved inequality* (5) *in [15] and [16] by an alternative method.*

(ii) *The above proofs generalize the direct proof of the usual Hardy inequality*

$$\int_\Omega \frac{|v(x)|^2}{r^2}dx \leqq \left(\frac{2}{N-2}\right)^2 \int_\Omega |v_r(x)|^2 dx, \qquad (12)$$

provided in the footnote of the textbook by Mizohata [10].

We can generalize Mizohata's proof as follows.

Lemma 2.1. *Assume* $N \geqq 1$ *and* $f = f(r)$*,* $g = g(r) \in C^1$*. Then for any* $v \in C_0^\infty$*, it holds that*

$$||v||_\varphi^2 \leqq ||v_r||_{f^2}^2,$$

where

$$\varphi = -\frac{N-1}{r}fg - (fg)_r - g^2.$$

[Proof.] Direct computations give

$$
\begin{aligned}
0 &\leqq |fv_r - gv|^2 = f^2 v_r^2 - 2fgv_r v + g^2 v^2 \\
&= f^2 v_r^2 - \nabla \cdot \left(fgv^2 \frac{x}{r} \right) + \frac{N-1}{r} fgv^2 + (fg)_r v^2 + g^2 v^2.
\end{aligned}
$$

Integrating both-sides of this equation by parts, we obtain the desired result. □

[An elementary proof of (12).] Choosing $f = 1$ and $g = ar^{-1}$ for some a, we can easily verify that $\varphi = -a\{a - (N-2)\} r^{-2}$. Thus, choosing $a = (N-2)/2$, we retrieve the usual Hardy inequality (12). □

As another application, consider the two–dimensional Hardy inequality:

Corollary 2.1. *Assume $N = 2$. Then for any $v \in C_0^\infty$, the following inequalities hold:*

$$
\int_\Omega \frac{|v(x)|^2}{r^{2+\delta}} dx \leqq \frac{4}{\delta^2 r_0^\delta} \int_\Omega |v_r(x)|^2 dx, \qquad (13)
$$

$$
\int_\Omega \frac{|v(x)|^2}{r^2 \{\log(Rr)\}} dx \leqq 4 \int_\Omega |v_r(x)|^2 dx, \qquad (14)
$$

where Ω denotes the exterior domain of $\mathbb{R}^2$, $\delta > 0$ is some constant, $r_0 =$ dist$(x, \partial\Omega)$, R is a number satisfying $Rr > 1$.

Remark 2.2. *Inequalities* (13) *and* (14) *are presented in Leis [8] and Dan-Shibata [2], respectively.*

Here, these proofs are simplified along with the proof of Lemma 2.2.

[Proof of Corollary 2.1.] Choosing $f(r) = r^{-\delta/2}$ and $g(r) = \dfrac{\delta}{2} r^{-1-\delta/2}$ with $\delta > 0$, we have $\varphi(r) = \dfrac{\delta^2}{4} r^{-2-\delta}$. Noting $r \geqq r_0$, (13) follows.

Choosing as $f(r) = 1$ and $g(r) = \dfrac{1}{2r\log(Rr)}$, we find $\varphi(r) = \dfrac{1}{4r^2\{\log(Rr)\}^2}$ which yields (14). □

Acknowledgments. The author would like to thank the referees and editor for their useful advice.

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(Received 11 June, 2013)
(Accepted 15 August, 2013)